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# The force on a small sphere in slow viscous flow

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The force on a small sphere translating relative to a slow viscous flow is found to $O(Re^{\frac{1}{2}})$ for two different fluid flows far from the sphere, namely pure rotation and pure shear. For pure rotation, the $O(Re^{\frac{1}{2}})$ correction to the Stokes drag consists of an increase in the drag. For pure shear, the $O(Re^{\frac{1}{2}})$ force contains a component perpendicular to the Stokes drag force.

## 1. Introduction

The force on an object immersed in a fluid flow is important in many practical situations. Soo's (1967) book gives an excellent discussion of many applications.

Happel & Brenner (1965) gave a thorough discussion of the force on a particle in Stokes flow, where fluid inertia is unimportant. Childress (1964) included the effect of inertia in the slow flow round a sphere in the situation where the fluid is rotating and the sphere translates along the axis of rotation. The correction to the Stokes drag was found to enter at $O(Re_{\Omega}^{\frac{1}{2}})$ for small $Re_{\Omega}$. Here $Re_{\Omega} = \rho\Omega a^2/\mu$, where $\rho$ is the fluid density, $\mu$ the viscosity, $\Omega$ the fluid rotation rate and $a$ the sphere radius. The translation velocity $V$ was also assumed to be small, the restriction being expressed by

$$Re_T \ll Re_{\Omega}^{\frac{1}{2}} \ll 1, \tag{1}$$

where $Re_T = \rho Va/\mu$.

Saffman (1965) calculated the effect of fluid inertia for a different situation: a small sphere translating slowly through a fluid undergoing parallel shear flow far from the sphere. He considered only sphere translations parallel to the fluid flow. Saffman's flow is restricted in the same manner as that of Childress, i.e. by (1), where now $\Omega \equiv \frac{1}{2}dU/dz$, in which $U$ is the fluid velocity far from the sphere. Harper & Chang (1968) generalized Saffman's calculation to bodies of arbitrary shape and more general sphere translation directions. They still considered parallel shear flow, however.

More recently, Herron, Davis & Bretherton (1975) have calculated the transverse force on a small sphere under centrifuge conditions, i.e. moving slowly relative to a rapidly rotating fluid. They assumed that

$$V/\Omega a \ll Re_{\Omega}^{\frac{1}{2}} \ll 1. \tag{2}$$

Thus their results are valid for $Re_{\Omega}^{\frac{1}{2}} \gg Re_T$.

The present work considers the force on a small sphere translating with respect to a fluid which is undergoing one of two different motions far from the sphere, namely pure shear or pure rotation. Our analysis follows the technique of Saffman and our results are valid for $Re_T \ll Re_{\Omega}^{\frac{1}{2}} \ll 1$, where $Re_{\Omega}$ is defined in the text. Our result for pure rotation differs from that of Childress in that our sphere motion is perpendicular to the rotation axis. Our result differs from that of Herron *et al.* in that our rotation is slow while theirs is fast, in the sense that we look at motions subject to (1) rather than (2). Consideration of the case of pure shear far from the sphere was motivated by the desire to include the single-sphere lift force in models of the mean motion of a particle–fluid mixture. Consideration of the problem from a rational mechanical point of view, subject to some assumptions about the nature of the two-phase mixture (Drew 1976), suggests that the lift force per unit mixture volume should be of the form $L\mathbf{V}\,.\,\mathbf{D}_f$, where $\mathbf{V}$ is the slip velocity and $\mathbf{D}_f$ is the mean fluid deformation tensor, defined to be $\mathbf{D}_f = \frac{1}{2}(\nabla\mathbf{v}_f + \mathbf{v}_f\nabla)$. Here $\mathbf{v}_f$ is the mean fluid velocity and $L$ is a scalar coefficient which depends on the Euclidean norm of $\mathbf{D}_f$.

## 2. Equations of motion

Let us calculate the force on a sphere which is not translating with respect to a slow viscous flow. The equations of motion are

$$\nabla^2\mathbf{q} - \nabla p = Re\,\mathbf{q}\,.\,\nabla\mathbf{q}, \quad \nabla\,.\,\mathbf{q} = 0 \tag{3}$$

with the boundary conditions

$$\mathbf{q} \to \mathbf{U} = \tfrac{1}{2}\kappa(z\mathbf{e}_1 + jx\mathbf{e}_3) + \cos\theta\,\mathbf{e}_1 + \sin\theta\mathbf{e}_3 \quad \text{as} \quad r\to\infty, \tag{4}$$

$$\mathbf{q} = \mathbf{\Omega}\times\mathbf{r} \quad \text{on} \quad r = 1. \tag{5}$$

Here $\mathbf{q}$ is the dimensionless velocity field, defined by $\mathbf{q} = \hat{\mathbf{q}}/V$, where $\hat{\mathbf{q}}$ is the fluid velocity; $p$ is the pressure; $\mathbf{r} = (x, y, z)$ is the dimensionless position vector, defined by $\mathbf{r} = \hat{\mathbf{r}}/a$, where $\hat{\mathbf{r}}$ is the position; $Re = Va/\nu$ is the Reynolds number based on the sphere radius and the slip velocity, where $V$ is the speed of the particle relative to the shear flow; $\theta$ is the angle the particle translation velocity makes with the $x$ axis; $a$ is the particle radius; $\nu$ is the kinematic viscosity; $\kappa$ is a dimensionless measure of the shearing, and $\mathbf{\Omega}$ is the rotation rate of the particle. The actual shear is given by

$$\hat{\mathbf{U}} = V\mathbf{U} = (\kappa V/2a)\,(z\mathbf{e}_1 + jx\mathbf{e}_3) + V(\cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_3).$$

The quantity $j$ will be taken to be $\pm 1$ for pure shear and pure rotation respectively. We note that $Re_T = Re$ and $Re_\Omega = \kappa Re$.

## 3. Inner expansion

We assume $Re \ll (Re\,\kappa)^{\frac{1}{2}} \ll 1$. Near the sphere we can formally expand the fields as

$$\mathbf{q}(\mathbf{r}) = \mathbf{q}^{(0)}(\mathbf{r}) + Re^{\frac{1}{2}}\mathbf{q}^{(1)}(\mathbf{r}) + \ldots, \tag{6}$$

$$p(\mathbf{r}) = p^{(0)}(\mathbf{r}) + Re^{\frac{1}{2}}p^{(1)}(\mathbf{r}) + \ldots, \tag{7}$$

where the terms not given explicitly are of higher order in $Re$. The two lowest-order fields $(\mathbf{q}^{(0)}, p^{(0)})$ and $(\mathbf{q}^{(1)}, p^{(1)})$ satisfy

$$-\nabla p^{(i)} + \nabla^2\mathbf{q}^{(i)} = 0, \quad \nabla\,.\,\mathbf{q}^{(i)} = 0 \quad (i = 0, 1). \tag{8}$$

The complete solution of (6) for $i = 0$ satisfying the scaled boundary conditions $\mathbf{q}^{(0)} = \mathbf{\Omega}\times\mathbf{r}$ on $|\mathbf{r}| = 1$ and $p^{(0)} \to 0$, $\mathbf{q}^{(0)} \to \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_3$ as $r \to \infty$ is given by

$$\mathbf{q}^{(0)} = \mathbf{U}^{(0)} + \sum_n \frac{\mathbf{r}}{r^{2n+3}}\left[\tfrac{1}{2}r^2 p_n^{(0)} - (2n+1)\,\phi_n^{(0)}\right] - \frac{n-2}{2n(2n-1)}\frac{\nabla p_n^{(0)}}{r^{2n-1}} + \frac{\nabla\phi_n^{(0)}}{r^{2n+1}} + \frac{\nabla\chi_n^{(0)}\times\mathbf{r}}{r^{2n+1}}, \tag{9}$$

$$p^{(0)} = \sum_n \frac{p_n^{(0)}}{r^{2n+1}}, \tag{10}$$

where

$$\mathbf{U}^{(0)} = \mathbf{U}, \tag{11a}$$

$$p_1^{(0)} = -\tfrac{3}{2}(x\cos\theta + z\sin\theta), \tag{11b}$$

$$p_2^{(0)} = \begin{cases} -5\kappa xz & (j=1), \\ 0 & (j=-1), \end{cases} \tag{11c}$$

$$\phi_1^{(0)} = -\tfrac{1}{4}(x\cos\theta + z\sin\theta), \tag{11d}$$

$$\phi_2^{(0)} = \begin{cases} -\tfrac{1}{2}\kappa xz & (j=1), \\ 0 & (j=-1), \end{cases} \tag{11e}$$

$$\chi_1^{(0)} = \begin{cases} \Omega y & (j=1), \\ (\Omega - \tfrac{1}{2}\kappa)\,y & (j=-1). \end{cases} \tag{11f}$$

The fields of next order ($i = 1$) are chosen in the same way, $\mathbf{U}^{(1)}$ being the mean flow velocity induced at the particle by the far-field Oseen flow.

To $O(Re^{\frac{1}{2}})$, the force and torque on the sphere can be computed from

$$\left.\begin{aligned} \mathbf{F} &= \mathbf{F}^{(0)} + Re^{\frac{1}{2}}\mathbf{F}^{(1)} + \dots, \\ \mathbf{M} &= \mathbf{M}^{(0)} + Re^{\frac{1}{2}}\mathbf{M}^{(1)} + \dots, \end{aligned}\right\} \tag{12}$$

where

$$\mathbf{F}^{(0)} = 6\pi\mathbf{U}^{(0)}|_{\mathbf{r}=0} = 6\pi(\mathbf{e}_1\cos\theta + \mathbf{e}_3\sin\theta), \tag{13a}$$

$$\mathbf{F}^{(1)} = 6\pi\mathbf{U}^{(1)}|_{\mathbf{r}=0}, \tag{13b}$$

$$\mathbf{M}^{(0)} = \begin{cases} -8\pi\Omega\mathbf{e}_2 & (j=1), \\ 8\pi(\tfrac{1}{2}\kappa - \Omega)\,\mathbf{e}_2 & (j=-1). \end{cases} \tag{13c}$$

In the absence of an applied torque,

$$\Omega = \begin{cases} 0 & (j=1), \\ \tfrac{1}{2}\kappa & (j=-1). \end{cases}$$

## 4. Outer expansions

Now let us consider the outer expansion. Let the strained co-ordinate $\mathbf{r}'$ be defined by

$$\mathbf{r}' = (\tfrac{1}{2}Re\,\kappa)^{\frac{1}{2}}\,\mathbf{r} \tag{14}$$

and assume solutions of the form

$$\mathbf{q} - \tfrac{1}{2}\kappa[z\mathbf{e}_1 + jx\mathbf{e}_3] - 6\pi\mathbf{S} = (\tfrac{1}{2}Re\,\kappa)^{\frac{1}{2}}\,\mathbf{q}'(\mathbf{r}'), \tag{15}$$

$$p - p_s(\mathbf{r}) = (\tfrac{1}{2}Re\,\kappa)\,p', \tag{16}$$

where $\mathbf{S}$ is the unit Stokeslet, defined by

$$\mathbf{S} = \frac{1}{8\pi}\left[(x\cos\theta + y\sin\theta)\,\nabla\left(\frac{1}{r}\right) - \frac{\mathbf{e}_1\cos\theta + \mathbf{e}_3\sin\theta}{r}\right], \tag{17}$$

and $p_s(\mathbf{r}) = -3(x\cos\theta + z\sin\theta)/2r^3$ is the most singular part of the inner pressure field.
The equations for $p'$ and $\mathbf{q}'$ to lowest order are

$$\nabla'^2\mathbf{q}' - \nabla' p' + (z'\,\partial\mathbf{q}'/\partial x' + jx'\partial\mathbf{q}'/\partial z' + jq_3'\mathbf{e}_1 + q_1'\mathbf{e}_3) = -6\pi(\mathbf{e}_1\cos\theta + \mathbf{e}_3\sin\theta)\,\delta(\mathbf{r}'), \quad (18)$$

$$\nabla'.\mathbf{q}' = 0. \quad (19)$$

Now let us compute $\mathbf{q}'$ using Fourier transforms. We define

$$\mathbf{\Gamma}(\mathbf{k}) = \frac{1}{8\pi^3}\int \mathbf{q}'\exp(-i\mathbf{k}.\mathbf{r}')\,d\mathbf{r}', \quad (20a)$$

$$\mathbf{q}'(\mathbf{r}') = \int \mathbf{\Gamma}(\mathbf{k})\exp(i\mathbf{k}.\mathbf{r}')\,d\mathbf{k}. \quad (20b)$$

The equations for $\mathbf{\Gamma}(\mathbf{k})$ are

$$-jk_1\frac{\partial\mathbf{\Gamma}}{\partial k_3} - k_3\frac{\partial\mathbf{\Gamma}}{\partial k_1} + j\Gamma_3\mathbf{e}_1 + \Gamma_1\mathbf{e}_3 - i\mathbf{k}\Pi + k^2\mathbf{\Gamma} = -\frac{3}{4\pi^2}(\mathbf{e}_1\cos\theta + \mathbf{e}_3\sin\theta), \quad (21)$$

$$\mathbf{k}.\mathbf{\Gamma} = 0.$$

Using the relation $\mathbf{k}.\partial\mathbf{\Gamma}/\partial k_i = -\Gamma_i$ $(i = 1,2,3)$, we can eliminate $\Pi$, the transform of the pressure, to get

$$\begin{aligned}-jk_1\frac{\partial\mathbf{\Gamma}}{\partial k_3} - k_3\frac{\partial\mathbf{\Gamma}}{\partial k_1} + j\Gamma_3\mathbf{e}_1 + \Gamma_1\mathbf{e}_3 - \frac{2\mathbf{k}}{k^2}(jk_1\Gamma_3 + k_3\Gamma_1) + k^2\mathbf{\Gamma}\\ = -\frac{3}{4\pi^2}\left\{\left[\left(1-\frac{k_1^2}{k^2}\right)\cos\theta - \frac{k_1k_3}{k^2}\sin\theta\right]\mathbf{e}_1\right.\\ + \left[-\frac{k_1k_3}{k^2}\cos\theta + \left(1-\frac{k_3^2}{k^2}\right)\sin\theta\right]\mathbf{e}_3\\ \left. + \left[\frac{k_1k_2}{k^2}\cos\theta + \frac{k_2k_3}{k^2}\sin\theta\right]\mathbf{e}_2\right\}.\end{aligned} \quad (22)$$

## 5. Pure rotation

For $j = -1$, the equations for $(\Gamma_1, \Gamma_3)$ are

$$k_1\frac{\partial\Gamma_1}{\partial k_3} - k_3\frac{\partial\Gamma_1}{\partial k_1} - \left(1-\frac{2k_1^2}{k^2}\right)\Gamma_3 + \left(k^2 - \frac{2k_1k_3}{k^2}\right)\Gamma_1 = -\frac{3}{4\pi^2}\left[\left(1-\frac{k_1^2}{k^2}\right)\cos\theta + \frac{k_1k_3}{k^2}\sin\theta\right], \quad (23)$$

$$k_1\frac{\partial\Gamma_3}{\partial k_3} - k_3\frac{\partial\Gamma_3}{\partial k_1} + \left(1-\frac{2k_3^2}{k^2}\right)\Gamma_1 + \left(k^2 + \frac{2k_1k_3}{k^2}\right)\Gamma_3 = -\frac{3}{4\pi^2}\left[-\frac{k_1k_3}{k^2}\cos\theta + \left(1-\frac{k_3^2}{k^2}\right)\sin\theta\right] \quad (24)$$

and $\Gamma_2$ can be found by quadrature.
We note that

$$\begin{aligned}\mathbf{\Gamma}(k_1,k_2,k_3) = (3/4\pi^2)\{[\Gamma_1^{(1)}(k_1,k_2,k_3)\cos\theta - \Gamma_3^{(1)}(-k_3,k_2,k_1)\sin\theta]\,\mathbf{e}_1\\ + [\Gamma_3^{(1)}(k_1,k_2,k_3)\cos\theta + \Gamma_1^{(1)}(-k_3,k_2,k_1)\sin\theta]\,\mathbf{e}_3\},\end{aligned} \quad (25)$$

where $(\Gamma_1^{(1)}, \Gamma_3^{(1)})$ is the solution of

$$k_1 \frac{\partial \Gamma_1^{(1)}}{\partial k_3} - k_3 \frac{\partial \Gamma_1^{(1)}}{\partial k_1} - \left(1 - \frac{2k_1^2}{k^2}\right) \Gamma_3^{(1)} + \left(k^2 - \frac{2k_1 k_3}{k^2}\right) \Gamma_1^{(1)} = -\left(1 - \frac{k_1^2}{k^2}\right), \tag{26}$$

$$k_1 \frac{\partial \Gamma_3^{(1)}}{\partial k_3} - k_3 \frac{\partial \Gamma_3^{(1)}}{\partial k_1} + \left(1 - \frac{2k_3^2}{k^2}\right) \Gamma_1^{(1)} + \left(k^2 + \frac{2k_1 k_3}{k^2}\right) \Gamma_3^{(1)} = \frac{k_1 k_3}{k^2}. \tag{27}$$

Let us introduce the characteristic co-ordinates

$$k_3 = \hat{k} \cos \sigma, \quad k_1 = -\hat{k} \sin \sigma, \tag{28}$$

where $k^2 = \hat{k}^2 + k_2^2$ is independent of $\sigma$. By considering a sequence of changes of the dependent variables, $(\Gamma_1^{(1)}, \Gamma_3^{(1)})$ can be found to be

$$\Gamma_1^{(1)} = -\frac{1}{k^2}\Big\{k^2[(k^2 - kk_2)\cos^2\sigma - (k_2^2 - kk_2)\sin^2\sigma](K_+^{-1} + K_-^{-1}) + \hat{k}^2 \cos\sigma \sin\sigma \left[\left(1 + \frac{2k_2}{k}\right) K_+^{-1} + \left(1 - \frac{2k_2}{k}\right) K_-^{-1}\right]\Big\}, \tag{29 a}$$

$$\Gamma_3^{(1)} = -\frac{1}{k^2}\Big\{[(k_2^2 - kk_2)\cos^2\sigma - (k^2 - kk_2)\sin^2\sigma]\left[-\left(1 + \frac{2k_2}{k}\right) K_+^{-1} - \left(1 - \frac{2k_2}{k}\right) K_-^{-1}\right] + k^2 \hat{k}^2 \sin\sigma \cos\sigma (K_+^{-1} + K_-^{-1})\Big\}, \tag{29 b}$$

where $K_\pm = k^4 + (1 \pm 2k_2/k)^2$.

The Stokeslet contribution is

$$\mathbf{\Gamma}_s^{(1)} = \left(-\frac{k_2^2 + \hat{k}^2 \cos^2\sigma}{2k^4}, -\frac{\hat{k}^2 \sin\sigma \cos\sigma}{2k^4}\right). \tag{30}$$

To compute the $O(Re^{\frac{1}{2}})$ force on the particle, we need to determine only the matching condition on $\mathbf{U}^{(1)}|_{\mathbf{r}=0}$ in the inner flow. This can be computed from $\mathbf{\Gamma}$ by subtracting the Stokeslet contribution and evaluating the velocity at $\mathbf{r}' = 0$. Thus we must compute

$$\int (\mathbf{\Gamma}^{(1)} - \mathbf{\Gamma}_s^{(1)})\, d\mathbf{k}. \tag{31}$$

This integral can be evaluated exactly by changing to spherical co-ordinates $(k, \delta, \sigma)$, where $\hat{k}^2 = k_1^2 + k_3^2$, $\hat{k} = k\cos\delta$ and $k_2 = k \sin\delta$. We then have

$$\int (\Gamma_3^{(1)} - \Gamma_{3s}^{(1)})\, d\mathbf{k} = 0, \tag{32}$$

$$\int (\Gamma_2^{(1)} - \Gamma_{2s}^{(1)})\, d\mathbf{k} = 0 \tag{33}$$

and

$$\int \Gamma_1^{(1)} - \Gamma_{1s}^{(1)} d\mathbf{k} = \frac{3(1 + 111\sqrt{3})}{140\sqrt{2}} \simeq 0{\cdot}976. \tag{34}$$

Now

$$\mathbf{F}^{(1)} = 6\pi \mathbf{U}^{(1)}, \tag{35}$$

so that

$$\mathbf{F} = 6\pi\mu a \mathbf{U}(1 + 0{\cdot}976(\tfrac{1}{2}\kappa Re)^{\frac{1}{2}}). \tag{36}$$

Thus the drag is increased by the rotation of the fluid. We note that the transverse force on the particle is zero.

## 6. Pure shear

For $j = 1$, the equations for $(\Gamma_1, \Gamma_3)$ are

$$-k_1\frac{\partial\Gamma_1}{\partial k_3}-k_3\frac{\partial\Gamma_1}{\partial k_1}+\left(1-\frac{2k_1^2}{k^2}\right)\Gamma_3+\left(k^2-\frac{2k_1k_3}{k^2}\right)\Gamma_1 = -\frac{3}{4\pi^2}\left[\left(1-\frac{k_1^2}{k^2}\right)\cos\theta-\frac{k_1k_3}{k^2}\sin\theta\right], \quad (37)$$

$$-k_1\frac{\partial\Gamma_3}{\partial k_3}-k_3\frac{\partial\Gamma_3}{\partial k_1}+\left(1-\frac{2k_3^2}{k^2}\right)\Gamma_1+\left(k^2-\frac{2k_1k_3}{k^2}\right)\Gamma_3 = -\frac{3}{4\pi^2}\left[\frac{-k_1k_3}{k^2}\cos\theta+\left(1-\frac{k_3^2}{k^2}\right)\sin\theta\right]. \quad (38)$$

Again, $\Gamma_2$ can be found by quadrature. We further note that

$$\mathbf{\Gamma}(k_1,k_2,k_3) = (3/4\pi^2)\{[\Gamma_1^{(1)}(k_1,k_2,k_3)\cos\theta+\Gamma_3^{(1)}(k_3,k_2,k_1)\sin\theta]\,\mathbf{e}_1 + [\Gamma_3^{(0)}(k_1,k_2,k_3)\cos\theta+\Gamma_1^{(0)}(k_3,k_2,k_1)\sin\theta]\,\mathbf{e}_3\}, \quad (39)$$

where $(\Gamma_1^{(0)}, \Gamma_3^{(0)})$ is the solution of

$$-k_1\frac{\partial\Gamma_1^{(0)}}{\partial k_3}-k_3\frac{\partial\Gamma_1^{(0)}}{\partial k_1}+\left(1-\frac{2k_1^2}{k^2}\right)\Gamma_3^{(0)}+\left(k^2-\frac{2k_1k_3}{k^2}\right)\Gamma_1^{(0)} = -\left(1-\frac{k_1^2}{k^2}\right), \quad (40)$$

$$-k_1\frac{\partial\Gamma_3^{(0)}}{\partial k_3}-k_3\frac{\partial\Gamma_3^{(0)}}{\partial k_1}+\left(1-\frac{2k_3^2}{k^2}\right)\Gamma_1^{(0)}+\left(k^2-\frac{2k_1k_3}{k^2}\right)\Gamma_3^{(0)} = \frac{k_1k_3}{k^2}. \quad (41)$$

We introduce characteristic co-ordinates in four different regions in $k_1, k_3$ space:

$$\begin{aligned}
&(\mathrm{I})\ k_1 = \hat{k}\sinh\sigma, \quad k_3 = \hat{k}\cosh\sigma \quad (\hat{k}>0),\\
&(\mathrm{II})\ k_1 = \hat{k}\cosh\sigma, \quad k_3 = \hat{k}\sinh\sigma \quad (\hat{k}>0),\\
&(\mathrm{III})\ k_1 = \hat{k}\sinh\sigma, \quad k_3 = \hat{k}\cosh\sigma \quad (\hat{k}<0),\\
&(\mathrm{IV})\ k_1 = \hat{k}\cosh\sigma, \quad k_3 = \hat{k}\sinh\sigma \quad (\hat{k}<0).
\end{aligned}$$

The solution can again be found after several transformations of the dependent variable. In region I, we have

$$\mathbf{\Gamma}^{(0)}(\sigma) = \int_\infty^\sigma \frac{\exp\left[k_2^2(\sigma-s)+\frac{1}{2}k^2(\sinh 2\sigma-\sinh 2s)\right]}{k^2(\sigma)}\mathbf{F}(s,\sigma)\,ds, \quad (42)$$

where

$$\mathbf{F}(s,\sigma) = \begin{bmatrix}(k_2^2+\hat{k}^2)\cosh s\cosh\sigma-k_2^2\sinh s\sinh\sigma\\(k_2^2-\hat{k}^2)\sinh\sigma\cosh s-k_2^2\cosh\sigma\sinh s\end{bmatrix}. \quad (43)$$

In II, a similar sequence of computations leads to

$$\mathbf{\Gamma}^{(0)}(\sigma) = \int_\infty^\sigma \frac{\exp\left[k_2^2(\sigma-s)+\frac{1}{2}k^2(\sinh 2\sigma-\sinh 2s)\right]}{k^2(\sigma)}\mathbf{G}(s,\sigma)\,ds, \quad (44)$$

where

$$\mathbf{G}(s,\sigma) = \begin{bmatrix}k_2^2\cosh\sigma\cosh s-(k_2^2-\hat{k}^2)\sinh\sigma\sinh s\\k_2^2\sinh\sigma\cosh s-(k_2^2+\hat{k}^2)\sinh s\cosh\sigma\end{bmatrix}. \quad (45)$$

The solution in III is exactly the same as that in I with $\hat{k}$ negative, and the solution in IV is exactly the same as that in II, with $\hat{k}$ negative.

To compute the force on the sphere, we must compute

$$\mathbf{I} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{\Gamma}^{(0)} - \mathbf{\Gamma}_s^{(0)}\, d\mathbf{k}, \tag{46}$$

where $\mathbf{\Gamma}_s^{(0)}$ is the Stokeslet contribution to the velocity transform:

$$\mathbf{\Gamma}_{s\,\mathrm{I,II}}^{(0)} = -k^{-4}\mathbf{f}_{\mathrm{I,II}}(\sigma), \tag{47}$$

where

$$f_{\mathrm{I}}(\sigma) = (k_2^2 + \hat{k}^2\cosh^2\sigma,\ -\hat{k}^2\cosh\sigma\sinh\sigma), \tag{48}$$

$$f_{\mathrm{II}}(\sigma) = (k_2^2 + \hat{k}^2\sinh^2\sigma,\ -\hat{k}^2\cosh\sigma\sinh\sigma). \tag{49}$$

Exploiting the symmetries, integrating by parts to take care of the Stokeslet contribution, and evaluating the integral with respect to the radial co-ordinate in $\mathbf{k}$ space, the integral can be reduced to

$$\mathbf{I} = 2\pi^{\frac{1}{2}}\int_0^1\int_{-\infty}^{\infty}\int_{\infty}^{\sigma} \frac{\mathbf{A}(s,\sigma,x)\,ds\,d\sigma\,dx}{[x^2(s-\sigma)+\frac{1}{2}(1-x^2)(\sinh 2s - \sinh 2\sigma)]^{\frac{1}{2}}\,K(\sigma,x)\,K^2(s,x)}, \tag{50}$$

where

$$K(s,x) = x^2 + (1-x^2)\cosh 2s \tag{51}$$

and

$$\mathbf{A}(s,\sigma,x) = \begin{bmatrix} (x^2-1)(2\sinh 2s)[(x^2+1)\cosh s\cosh\sigma - (3x^2-1)\sinh s\sinh\sigma] \\ \quad +[x^2+(1-x^2)\cosh 2s][(x^2+1)\sinh s\cosh\sigma - (3x^2-1) \\ \quad \times\cosh s\sinh\sigma] \\ (x^2-1)(2\sinh 2s)[(1+x^2)\cosh s\sinh\sigma - (3x^2-1)\sinh s\cosh\sigma] \\ \quad +[x^2+(1-x^2)\cosh 2s][(x^2+1)\sinh s\sinh\sigma - (3x^2-1) \\ \quad \times\cosh s\cosh\sigma] \end{bmatrix}. \tag{52}$$

The integral (50) has been approximated numerically to get

$$\mathbf{I} \simeq 2\pi^{\frac{1}{2}}(0{\cdot}37,\ 1{\cdot}86). \tag{53}$$

Thus the force on the particle is given by

$$\mathbf{F} = 6\pi\mu a\mathbf{U}\,.\,\{[1+0{\cdot}10(\tfrac{1}{2}\kappa Re)^{\frac{1}{2}}](\mathbf{e}_1\mathbf{e}_1+\mathbf{e}_3\mathbf{e}_3) + 0{\cdot}502(\tfrac{1}{2}\kappa Re)^{\frac{1}{2}}(\mathbf{e}_1\mathbf{e}_3+\mathbf{e}_3\mathbf{e}_1)\}. \tag{54}$$

The first part represents the Stokes drag plus a shearing modification. We note that the shearing increases the drag. The second part is a force with a component perpendicular to the drag, and hence represents a lift force.

## 7. Conclusion

We have computed the force on a sphere in slow viscous flow where the motion far from the sphere is (i) pure rotation and (ii) pure shear. For the case of pure rotation, we found that the interaction of drag and deformation leads to an increase in the drag to lowest order and that the sphere experiences no transverse force to this order. For the case of pure shear, we found that the interaction of drag and deformation leads to an $O(Re_\Omega^{\frac{1}{2}})$ modification to the drag and an $O(Re_\Omega^{\frac{1}{2}})$ transverse force.

We note that in situations where this $O(Re_\Omega^{\frac{1}{2}})$ force is valid (namely slow shear flow past a small sphere), this force is smaller than the Stokes drag force. This force, however, has a component perpendicular to the drag force. We therefore refer to this force as a lift force. Even though the lift force may be negligible in magnitude compared with the drag force, it can still have an appreciable effect in directions transverse to the slip.

The results in this paper, to lowest order, do not contradict the form of the lift force proposed by Drew (1976). If the lift force is $L\mathbf{V} . \mathbf{D}_f$, then $L = \alpha l \rho^{\frac{1}{2}} \mu^{\frac{1}{2}} a^{-1} |\mathbf{D}_f|^{-\frac{1}{2}}$, where $l \simeq 2{\cdot}2$ and $\alpha$ is the particle volume fraction. This value of $l$ is close to that derived by Drew from Saffman's (1965, 1968) result. The form of the lift matrix derived by Harper & Chang (1968), however, suggests that his result cannot be generalized to the more general flow, and that Drew's (1976) assumed form is incomplete.

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